THREE-DIMENSIONAL SELF-SIMILAR PROBLEMS INVOLVING THE PENETRATION OF BLUNT BODIES IN AN IDEAL COMPRESSIBLE FLUID (SUPERSONIC CASE)

F. M. Borodich

UDC 532.5:531.001.362

In a linear setting we consider the three-dimensional problem of the penetration of a blunt body into a half-space of ideal compressible fluid, where the rate of expansion of the boundary of the wetted surface of the body is greater than the speed of sound in the fluid (supersonic case).

It is well-known that the solution of such problems in a general setting is given by an expression involving a retarded potential [1-5]. In the present problem we identify a class of self-similar problems involving the penetration of blunt bodies into a compressible fluid. We show that the general formulations simplify in the case of axisymmetric self-similar problems. We supply the results of some numerical calculations. For the self-similar problem involving the penetration of a blunt cone the results agree with those obtained in [1, 4].

1. Statement of the Problem. Let us assume that a rigid blunt body penetrates with speed V(t) an ideal weightless weakly compressed fluid occupying, in a state of rest, the half-space $x_3 \geqslant 0$. The velocity of the body is assumed to be perpendicular to the plane $x_3 = 0$. It is assumed that V(t), over the whole duration of the operation, is small in comparison with a (the speed of sound in the fluid); in addition, it is assumed that the rate of expansion of the region of interaction of the body with the fluid at each point of its boundary is greater at an arbitrary time instant than the speed of sound (the rate of expansion is calculated along the normal to the boundary).

We take the origin of a Cartesian coordinate system at the point where the body first touches the free surface. Axis x_3 is directed downwards into the fluid, and axes x_1 and x_2 lie along the initial free surface.

The penetrating bodies are blunt, i.e., we assume that the angle between the tangent plane to the body and the plane $x_3 = 0$ is small over the whole interval of time of consideration of the problem; the depth of penetration is small and the solution of the penetration problem can be sought on the basis of the linearized equations of hydrodynamics [1-3]. The boundary conditions with the boundary of contact of the body with the fluid are referred to the plane $x_3 = 0$. The flow that arises is assumed to be potential flow.

The velocity of the fluid particles $v = \{v_1, v_2, v_3\}$ and the pressure p(x, t) are determined in terms of the potential Φ from the formulas [1-5]

$$v(\mathrm{x}, t) = \mathrm{grad}\, \Phi(\mathrm{x}, t), \; p = -\rho \partial\Phi(\mathrm{x}, t)/\partial t \tag{1.1}$$

(ρ is the initial density of the fluid). The penetration problem then reduces to finding the potential Φ(x, t) satisfying the wave equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{1}{a^2}\frac{\partial^2}{\partial t^2}\right)\Phi(\mathrm{x}, t) = 0 \quad \forall t > 0, \tag{1.2}$$

the boundary condition

$$\partial\Phi(x_1, x_2, 0, t)/\partial x_3 = V(t), \; (x_1, x_2) \in G(t) \tag{1.3}$$

[G(t) is the region of interaction of the body with the fluid], and the condition

$$\Phi \equiv 0 \tag{1.4}$$

on the front of the wave and before it.

The region of interaction G(t) in the supersonic case may be determined directly from the form of the surface of the penetrating body. In particular, if the surface of the blunt body

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 110-114, May-June, 1988. Original article submitted March 16, 1987.

0021-8944/88/2903-0415$12.50

is given by the function $x_3 = -f(x_1, x_2)$, then the points $(x_1^*(t), x_2^*(t))$, lying on the boundary $\partial G(t)$ of the region of interaction, satisfy at time t the condition

$$f(x_1^*, x_2^*) = \int_0^t V(\xi)\, d\xi. \tag{1.5}$$

2. Three-Dimensional Self-Similar Problems. General methods for the selection of similarity transformations, permitted by the differential equations, are presented in [6]. We consider below only one transformation, permitted by Eq. (1.2), namely, a transformation of stretching of the coordinates, i.e., self-similar problems. In this class of problems, for bodies whose surface is specified by a positive smooth homogeneous function of degree d, $(d \geqslant 1)$, we obtain the following similarity statement.

Statement 1. Let the form of the surface of the body be given by a function f such that

$$\begin{gathered}\forall\lambda \geqslant 0 \quad f(\lambda x_1, \lambda x_2) = \lambda^d f(x_1, x_2) \quad \forall (x_1, x_2) \in R^2,\\ f(x_1, x_2) \in C^1(R^2 \setminus \{0\}),\ f(x_1, x_2) > 0 \quad \forall (x_1, x_2) \in R^2 \setminus \{0\},\end{gathered} \tag{2.1}$$

and let the penetration velocity be a step function of the form

$$V(t) = V(1)t^{(d-1)} \tag{2.2}$$

[V(1) is a constant].

Then, if a solution of problem (1.2)-(1.5), specified by the potential Φ, is known at a time t_1, $(t_1 > 0)$, the solution at an arbitrary time t, $(t > 0)$, is determined from the similarity relationship

$$\Phi(\mathbf{x}, t) = \lambda^{-d}\Phi(\lambda\mathbf{x}, t_1) \quad (\lambda = t_1/t). \tag{2.3}$$

Moreover, the region G(t) is obtained from $G(t_1)$ by a homothetic transformation with center at the coordinate origin: $((x_1, x_2) \in G(t)) \Leftrightarrow ((\lambda x_1, \lambda x_2) \in G(t_1))$. This may be proved by directly verifying that the expression (2.3), with Eqs. (2.1) and (2.2) taken into account, satisfies the conditions (1.2)-(1.5). The verification is made in the same way as in the proof of similarity theorem in a three-dimensional contact problem of elasticity theory (see, for example, [7]).

COROLLARY. It follows from Statement 1 and conditions (1.1) that in the case under consideration the velocity vector and the pressure at a point of the fluid are determined from the similarity relations $v(\mathbf{x}, t) = \lambda^{(1-d)}v(\lambda\mathbf{x}, t_1)$, $p(\mathbf{x}, t) = \lambda^{(1-d)} p(\lambda\mathbf{x}, t_1)$.

Remark. A similar statement relating to similarity is also valid in the subsonic case of the problem under consideration.

3. Solution of Problems for Bodies of Revolution. It is a known fact [1, 3, 7] that the velocity potential in a problem involving a body of arbitrary shape can be written in the form

$$\Phi(\mathbf{x}, t) = -\frac{1}{2\pi}\iint_{R^2} \frac{\partial\Phi/\partial x_3(\xi, \eta, 0, t - r_1/a)}{r_1}\, d\xi\, d\eta, \tag{3.1}$$

where $r_1 = \sqrt{(x_1 - \xi)^2 + (x_2 - \eta)^2 + x_3^2}$. Moreover, the force F with which the body acts on the fluid is determined from the expression [5]

$$F(t) = \rho a S V(t) \tag{3.2}$$

[S is the area of the region G(t)]. We remark that the function $\partial\Phi/\partial x_3(\xi, \eta, 0, t - r_1/a)$ is known: if $(\xi, \eta) \in G(t - r_1/a)$, it is equal to $V(t - r_1/a)$; in the contrary case it is equal to zero. Thus, in principle, the problem is solved for a body of arbitrary shape. However, the solution in concrete cases encounters definite difficulties [1-4].

For a body of revolution in a cylindrical coordinate system r, θ, z($x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$) we find that the shape of the body at an arbitrary time satisfies the equation $z = -f(r)$.

Statement 2. Let a blunt body of revolution, whose shape may be described by an arbitrary monotonically increasing smooth function f(r), f(0) = 0, penetrate into an ideal compressible fluid half-space with velocity V(t). Here

$$V(t) = cf'(ct) \tag{3.3}$$

(c is a positive constant). Then the rate of propagation of the boundary $\partial G(t)$ of the region of interaction is constant and equal to c.

Proof. Let $r_*(t)$ be the radius of the boundary $\partial G(t)$ of the region of interaction. From Eqs. (1.5) and (3.3) we then have $f(r_*(t)) = \int_0^t V(\tau)\,d\tau = f(ct)$, from which we obtain $r_*(t) = ct$, i.e., Statement 2 is valid.

If we denote by M the ratio of the constant rate c of expansion of the radius r_* to the speed of sound a, we find $r_*(t) = \mathrm{M}at$, $\mathrm{M} = c/a$, $\mathrm{M} > 1$. If the body of revolution is determined by the power function $z = -Ar^d (A > 0)$ and the speed of the body by the expression (2.2), then $c = (V(1)/Ad)^{1/d}$.

For a body whose shape is described by a power function, the force of interaction with the fluid, taking relation (3.2) into account, is $F(t) = \pi a\rho (Ad)^{-(2/d)} V(1)^{(1+2/d)} t^{(d+1)}$. Expression (3.1) for the velocity potential assumes the form

$$\Phi(r, z, t) = -\frac{V(1)}{2\pi}\int_{\theta_1}^{\theta_2}\int_{\psi_1}^{\psi_2} \frac{(t - r_1/a)^{(d-1)}}{r_1}\,\psi\, d\psi\, d\theta, \tag{3.4}$$

where $r_1 = \sqrt{r^2 + z^2 + \psi^2 - 2r\psi\cos\theta}$; $\psi = \sqrt{\xi^2 + \eta^2}$; θ is the angle between the vectors $\mathbf{r}$ and $\boldsymbol{\psi}$.

For the limiting values $(\theta)_{1,2}$ and $(\psi)_{1,2}$ (similar to what was done in the self-similar problem involving a blunt cone [1, 4]), we obtain, taking relation (3.3) into account, the following: for $r^2 + z^2 \leqslant a^2t^2$

$$\theta_1 = 0,\ \theta_2 = 2\pi,\ \psi_1 = 0,$$

$$\psi_2 = -\frac{\mathrm{M}}{\mathrm{M}^2 - 1}\left[at - \mathrm{M}r\cos\theta - \sqrt{(at - \mathrm{M}r\cos\theta)^2 + (a^2t^2 - r^2 - z^2)(\mathrm{M}^2 - 1)}\right];$$

for $r^2 + z^2 > a^2t^2$

$$\psi_{1,2} = -\frac{\mathrm{M}^2}{(\mathrm{M}^2 - 1)}\left[at - \mathrm{M}r\cos\theta \pm \sqrt{(at - \mathrm{M}r\cos\theta)^2 + (a^2t^2 - r^2 - z^2)(\mathrm{M}^2 - 1)}\right],$$

$$\theta_1 = -\theta_*,\quad \theta_2 = \theta_*,\quad \theta_* = \arccos\left[\frac{at + \sqrt{(1 - \mathrm{M}^2)(a^2t^2 - r^2 - z^2)}}{\mathrm{M}r}\right].$$

We introduce the dimensionless variables $r_0, z_0, \psi_0, (\psi_*)_0, V_0, \Phi_0, p_0$, where $r_0 = r/(at)$, $z_0 = z/(at)$, $\psi_0 = \psi/(at)$, $(\psi_*)_0 = \psi_*/(at)$, $V_0 = V/a$, $\Phi_0 = \Phi/(a^2t)$, $p_0 = p/(\rho a^2 V_0)$. Henceforth we shall use only the dimensionless variables, omitting the subscript in doing so. We can then write expression (3.4) as

$$\Phi(r, z, t) = -\frac{V(1)}{2\pi}\int_{\theta_1}^{\theta_2}\int_{\psi_1}^{\psi_2} \frac{(1 - r_1)^{(d-1)}}{r_1}\,\psi\, d\psi\, d\theta. \tag{3.5}$$

For the case in which d is a positive integer we find from Eq. (3.5) that

$$\Phi = -\frac{V(t)}{2\pi}\left[\Phi_1 + \sum_{n=1}^{K}\frac{(d-1)(d-2)\ldots(d-2n)}{(2n)!}P_{2n-1} - \sum_{n=1}^{L}\frac{(d-1)\ldots(d-2n+1)}{(2n-1)!}P_{2n-2}\right],$$

where K = N − 1, L = N for $d = 2N$, $\forall N = 1, 2, \ldots$; $K = N$, $L = N + 1$ for $d = 2N + 1$, $\forall N = 0, 1, \ldots$. Here we have introduced the notation

$$\Phi_1 = \int_{\theta_1}^{\theta_2}\int_{\psi_1}^{\psi_2}\frac{\psi}{r_1}\,d\psi\, d\theta,\quad P_{2n-2} = \int_{\theta_1}^{\theta_2}\int_{\psi_1}^{\psi_2}(r_1^2)^{(n-1)}\psi\, d\psi\, d\theta,\ P_{2n-1} = \int_{\theta_1}^{\theta_2}\int_{\psi_1}^{\psi_2}(r_1)^{(2n-1)}\psi\, d\psi\, d\theta. \tag{3.6}$$

The inner integrals in expression (3.6) can be taken in another way.

As an example, we consider the case d = 2, i.e., we consider the problem of uniform acceleration of penetration into the fluid of a paraboloid. Then

$$\Phi = -\frac{V(t)}{2\pi}[\Phi_1 - P_0] = -\frac{V(t)}{2\pi}\left[\Phi_1 - \frac{1}{2}\int_{\theta_1}^{\theta_2}(\psi_2^2 - \psi_1^2)\,d\theta\right].$$

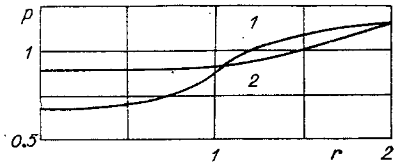

Fig. 1

The function Φ_1 can be brought to the form

$$\Phi_1 = \int\limits_{\theta_1}^{\theta_2} \left[r_1(\psi_2) - r_1(\psi_1) + r\cos\theta \ln \frac{\psi_2 - r\cos\theta + r_1(\psi_2)}{\psi_1 - r\cos\theta + r_1(\psi_1)} \right] d\theta,$$

which gives the solution of the problem of penetration of a blunt cone into a compressible fluid [1].

Figure 1 shows graphs of pressure distribution p over the wetted surface of a body in two self-similar problems for M = 2: curve 1 is for the penetration of a cone (d = 1), and curve 2 is for the penetration of a paraboloid (d = 2).

The author thanks A. G. Khovanskii for useful discussions of the present paper.

LITERATURE CITED

1. A. Ya. Sagomonyan and V. B. Poruchikov, Three-Dimensional Problems of Unsteady Motion of a Compressible Fluid [in Russian], Moscow State Univ. (1970).
2. É. I. Grigolyuk and A. G. Gorshkov, Interaction of Elastic Structures with a Fluid [in Russian], Sudostroenie, Leningrad (1976).
3. V. D. Kubenko, Penetration of Elastic Shells into a Compressible Fluid [in Russian], Naukova Dumka, Kiev (1981).
4. Sh. D. Shamgunov, "Impact of an arbitrary blunt body of revolution onto the surface of a compressible fluid half-space," in: Proceedings of All-Union Symposium on the Propagation of Elastic-Plastic Waves in a Continuous Medium [in Russian], Izd. Akad. Nauk AzSSR, Baku (1966).
5. R. Skalak and D. Feit, "Impact on the surface of a compressible fluid," Trans. ASME, B88, No. 3 (1966).
6. L. V. Ovsyannikov, Group Properties of Differential Equations [in Russian], Nauka, Moscow (1978).
7. F. M. Borodich, "On a problem of contact of two predistorted half-spaces," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1984).